The main theorem cont.

GOAL: $C^*_{\mathrm{Lie}} : \mathrm{Lie} \longrightarrow (\mathrm{CAlg}^{\mathrm{aug}})^{op}$ $: \mathfrak{D}$

(1) C^*_{Lie} has right adjoint $\mathfrak{D}$

use: adjoint functor theorem

$\longrightarrow$ need to show that C^*_{Lie} preserves small colims

(2) Then show: this adjoint is "almost" an equivalence,

$\Longrightarrow$ the equivalence Ψ we ultimately want.

$$\begin{tikzcd}
\mathrm{Lie} \arrow[rr, bend left, "\Psi"] \arrow[dr, "C^*_{\mathrm{Lie}}"] \arrow[ddr, bend right, "\Theta = \text{forget}"'] & & \mathrm{FMP} \arrow[ddl, bend left, "T = (-)\circ \mathrm{Dual}[-1]"] \\
& (\mathrm{CAlg}^{\mathrm{aug}})^{op} \arrow[ul, dashed, bend left, "\mathfrak{D}"] \arrow[ur, "\mathrm{Spf}"'] & \\
& \mathrm{Mod}_k &
\end{tikzcd}$$

(1) C^*_{Lie} preserves small colims.

$$\begin{tikzcd}
 & (\mathrm{CAlg}_k^{aug})^{op} \arrow[dr, "\text{forget}"] & \\
\mathrm{Lie}_k \arrow[ur, "C^*_{Lie}"] \arrow[dr, "C_*"'] & & (\mathrm{Mod}_k^{op})_{k/} \\
 & (\mathrm{Mod}_k)_{k/} \arrow[ur, "D"'] &
\end{tikzcd}$$

•) D induced by $\begin{cases} \mathrm{Vect}_k^{dg} \longrightarrow (\mathrm{Vect}_k^{dg})^{op} \\ V_* \longmapsto V_*^\vee, \end{cases}$

which sends homotopy colims to homotopy colims

$\overset{D}{\Longrightarrow}$ preserves small colims.

•) colimits in $(\mathrm{CAlg}_k^{aug})^{op}$ are computed in $(\mathrm{Mod}_k^{op})_{k/}$

$\Rightarrow$ C^*_{Lie} preserves small colims iff forget $\circ\, C^*_{Lie}$ does

$\Longrightarrow$ enough to show that $C_* : \mathrm{Lie}_k \longrightarrow (\mathrm{Mod}_k)_{k/}$ preserves small colims. $\boxed{\text{Use:}}$

(A)

<u>Prop:</u> $F: \mathcal{C} \longrightarrow \mathcal{D}$ preserves small colims iff

(combining HTT 4.2.3.11 + HA 1.3.3.10) F preserves finite coproducts + small sifted colimits

Recall: $C_*(\mathfrak{g}_*) \overset{\text{gr v.sp.}}{\cong} \mathrm{Sym}^*(\mathfrak{g}_*[1])$, $d(x_1 \cdots x_n) =$

$$\sum (-1)^{\cdots} x_1 \cdots x_{i-1}\, dx_i\, x_{i+1} \cdots x_n$$

$$\cancel{\sum (-1)^{\cdots} x_1 \cdots x_{i-1} x_{i+1} \cdots x_n}$$

$$+ \sum (-1)^{\cdots} x_1 \cdots x_{i-1} x_{i+1} \cdots x_{j-1} [x_i, x_j] x_{j+1} \cdots x_n$$

Filtration: $\mathrm{Sym}^{\le n}(\mathfrak{g}) \simeq \bigoplus_{i \le n} \mathrm{Sym}^i(\mathfrak{g})$

$$\rightsquigarrow \quad k \simeq C_*^{\le 0}(\mathfrak{g}) \hookrightarrow C_*^{\le 1}(\mathfrak{g}) \hookrightarrow C_*^{\le 2}(\mathfrak{g}) \hookrightarrow \cdots$$

$$\mathrm{Sym}^n(\underset{[1]\,?}{\mathfrak{g}}) \simeq C^{\le n}(\mathfrak{g}) / C^{\le n-1}(\mathfrak{g}) \quad \text{in dg Vect}$$

Prop 1 $f : \mathfrak{g}_* \to \mathfrak{g}'_*$ quasi-iso of dglas. Then
$C_*(f) : C_*(\mathfrak{g}_*) \to C_*(\mathfrak{g}'_*)$ is quasi-iso of chain cplxes

Pf: quasi-isos closed under filtered colimits
$\Rightarrow$ Show that $f_n : C_*^{\le n}(\mathfrak{g}_*) \to C_*^{\le n}(\mathfrak{g}'_*)$ is q-i. $\forall n \ge 0$

Use induction: $n = 0$ ✓ $\quad n > 0 \rightsquigarrow$ comm. diag of short ex. seq.

$$\begin{tikzcd}
0 \arrow[r] & C_*^{\le n-1}(\mathfrak{g}_*) \arrow[r] \arrow[d, "f_{n-1}"] & C_*^{\le n}(\mathfrak{g}_*) \arrow[r] \arrow[d, "f_n"] & \mathrm{Sym}^n(\mathfrak{g}[1]) \arrow[r] \arrow[d, "\phi"] & 0 \\
0 \arrow[r] & C_*^{\le n-1}(\mathfrak{g}'_*) \arrow[r] & C_*^{\le n-1}(\mathfrak{g}'_*) \arrow[r] & \mathrm{Sym}^n(\mathfrak{g}'_*[1]) \arrow[r] & 0
\end{tikzcd}$$

$\Rightarrow$ by ind. hypothesis, need to show ϕ is quasi-iso.
ϕ is retract of $\mathfrak{g}^{\otimes n}[n] \xrightarrow[\simeq]{f^{\otimes n}} \mathfrak{g}'^{\otimes n}[n] \Rightarrow$ quasi-iso. ∎

$\Rightarrow \quad C_* : \mathrm{Lie}_k \to (\mathrm{Mod}_k)_{k/}$ functor of ∞-cats.

Recall: want to show this preserves small colimits.

$\boxed{\text{Step 1}}$

Analyze on free dglas –

Prop 2 V_* dg vect. sp., $\mathfrak{g}_* = \mathrm{Free}_{Lie}(V_*)$. Then incl. of chain cplxes

$$\xi : k \oplus V_*[1] \hookrightarrow k \oplus \mathfrak{g}_*[1] \simeq C_*^{\leq 1}(\mathfrak{g}) \hookrightarrow C_*(\mathfrak{g}_*)$$

is a quasi-iso.

Recall from Dan + Matthew's talks:

if $A = \mathrm{Tens}(V) = \mathrm{Tens}(V)/(R)$, $R = 0$, $= \mathcal{U}(\mathrm{Free}_{Lie}(V))$ ← univ. env. alg.

$$A^! = \mathrm{Tens}(V^*)/(R^\perp) \xleftarrow{\ \sim\ } k \oplus V^*[-1]$$

$R^\perp$ "$= V^* \otimes V^*$"

$A^! = \underline{C^*_{Lie}(\mathrm{Free}(V))}$

$\left(\mathcal{U}\mathfrak{g}^! = C^{Lie}_*(\mathfrak{g}) \right)$

$\boxed{\text{Step 2}}$ reduce to free case by "resolving"

Claim: $\mathfrak{g}$ any dgla. Then $\mathfrak{g}$ is the geometric realization of a simplicial object $(\mathfrak{g}_*)$ in Lie_k s.t. each $\mathfrak{g}_n$ lies in the essential image of $\mathrm{Free} : \mathrm{Mod}_k \longrightarrow \mathrm{Lie}_k$.

Idea:

$$\begin{tikzcd} \mathrm{Alg} \arrow[d, bend right, "\text{forget}"'] \\ \mathrm{Mod}_k \arrow[u, bend right, "\text{Free} = \text{Tens}"'] \end{tikzcd} \qquad \ni A \quad \text{produce resolution}$$

$$\begin{tikzcd} & & \mathrm{Tens}(A) \arrow[r, two heads, "\varepsilon_A"] & A \\ & \mathrm{Tens}(\ker \varepsilon_A) \arrow[ur] \arrow[r, two heads, "\ker \varepsilon_2"] & \ker(\varepsilon_A) \arrow[u] & \\ \mathrm{Tens}(\ ) \arrow[ur] \arrow[r, two heads] & \ker \varepsilon_2 \arrow[u] & & \\ \cdots \arrow[ur] & & & \end{tikzcd}$$

free resolution of A. (huge!!)

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Similarly,

$$\begin{tikzcd} \mathrm{Lie}_k \arrow[d, "\theta = \mathrm{forget}"'] \\ \mathrm{Mod}_k \arrow[u, bend right, "\mathrm{Free}"'] \end{tikzcd}$$

θ is "monadic"

$T = \theta \circ \mathrm{Free}$ monad

$\rightsquigarrow$ can do monadic resolution in ∞-categorical setting.

Prop $C_*: \mathrm{Lie}_k \longrightarrow (\mathrm{Mod}_k)_{k/}$ preserves small colimits.

Pf: Show: (a) preserves small sifted colimits

(b) finite coproducts

(a) (small) sifted colims: (cf filtered colims + geometr. realizations)

enough:

$$\begin{tikzcd} \mathrm{Lie}_k \arrow[r, "C_*"] \arrow[rr, bend right] & (\mathrm{Mod}_k)_{k/} \arrow[r] & \mathrm{Mod}_k \end{tikzcd}$$

preserves sifted colims

Strategy similar to previous prop 1

$\forall n$: $C_*^{\leq n}: \mathrm{Lie}_k \longrightarrow \mathrm{Mod}_k$,

C_* is filtered colimit of these functors!

$\underset{\text{HTT 5.5.2.3}}{\Longrightarrow}$ It suffices to show that $C_*^{\leq n}$ preserve sifted colims.

Induction on n: $n \leq 0$ trivial.

induction step: Have fiber sequence of functors (of ∞-cats.)

$$C_*^{\leq n-1} \longrightarrow C_*^{\leq n} \longrightarrow \underbrace{\mathrm{Sym}_k^n}\circ\underbrace{\theta[1]}$$

preserves small sifted colims by induction hypothesis

is retract of functor $V_* \longmapsto V_*^{\otimes n}$

preserves (sifted) small colims

Here sifted is crucial: eg $n=2$: $(V,W) \xrightarrow{-\otimes-} V \otimes W$ pres. colims in each var.

$V \xmapsto{\mathrm{diag}} (V,V) \xmapsto{-\otimes-} V \otimes V$

eg $n=2$: $\quad (\mathrm{Vect}^{dg})^{\times 2} \xrightarrow[-\otimes-]{} \mathrm{Vect}^{dg}$

$$(V,W) \longmapsto V\otimes W$$ pres. colims in each variable

$$\mathrm{Mod}_k\ \cancel{\mathrm{Vect}^{dg}} \xrightarrow{\text{diag}} \Big(\overset{\mathrm{Mod}_k}{\cancel{\mathrm{Vect}^{dg}}}\Big)^{\times 2} \xrightarrow{-\otimes-} \overset{\mathrm{Mod}_k}{\cancel{\mathrm{Vect}^{dg}}}$$

$$V \longmapsto (V,V) \longmapsto V\otimes V$$

sifted colim:

$$\begin{tikzcd} \overset{\mathrm{Mod}_k}{\cancel{\mathrm{Vect}^{dg}}} \arrow[r, "\text{diag}"] & \Big(\overset{\mathrm{Mod}_k}{\cancel{\mathrm{Vect}^{dg}}}\Big)^{\times 2} \arrow[r, "-\otimes-"] & \overset{\mathrm{Mod}_k}{\cancel{\mathrm{Vect}^{dg}}} \\ K \arrow[u] \arrow[r, "\text{cofinal}"'] & K\times K \arrow[u] & \end{tikzcd}$$

sifted diagram

(b) <u>finite coproducts</u>:

- C_* preserves initial objects $\Longrightarrow$ enough to show that C_* preserves pairw. coprod, i.e. for

$C_* : \mathrm{Lie}_k \to \boxed{(\mathrm{Mod}_k)_{k/}}$

$$\mathfrak{g}''_* = \mathfrak{g}_* \amalg \mathfrak{g}'_*,$$

$$\begin{tikzcd} k \arrow[r] \arrow[d] & C_*(\mathfrak{g}_*) \arrow[d] \\ C_*(\mathfrak{g}'_*) \arrow[r] \arrow[ur, phantom, "\ulcorner"] & C_*(\mathfrak{g}''_*) \end{tikzcd}$$ is pushout. in Mod_k

Use "monadic" resolution:

$$\begin{tikzcd} \mathrm{Lie}_k \arrow[d, "\theta"'] \\ \mathrm{Mod}_k \arrow[u, bend right, "\mathrm{Free}"'] \end{tikzcd}$$

Write $\mathfrak{g}_* =$ geometric realization of simpl. object s.t. each $(\mathfrak{g}_*)_n$ is in essential image of Free

$\mathfrak{g}'_* = \quad$ — " — $\quad (\mathfrak{g}'_*)_\bullet$

$\mathfrak{g}''_* = \quad$ — " — $\quad (\mathfrak{g}''_*)_\bullet \qquad (\mathfrak{g}''_*)_n = (\mathfrak{g}'_*)_n \amalg (\mathfrak{g}_*)_n$

b/c geom. real commutes w/ coproducts

by (a), C_* commutes w/ geometr. realiz. (b/c are sifted colims)

$\Rightarrow$ enough to show that

$$\begin{tikzcd} k \arrow[r] \arrow[d] & C_*((\mathfrak{g}_*)_n) \arrow[d] \\ C_*((\mathfrak{g}'_*)_n) \arrow[r] & C_*((\mathfrak{g}''_*)_n) \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

is pushout in Mod_k.

Show for free $\Rightarrow$ $\mathfrak{g}_* \simeq \mathrm{Free}(V_*)$, $\mathfrak{g}'_* \simeq \mathrm{Free}(V'_*)$ $\Rightarrow$ $\mathfrak{g}''_* \simeq \mathrm{Free}(V_* \oplus V'_*)$

$\overset{\text{Prop 2}}{\Longrightarrow}$ enough to show

$$\begin{tikzcd} k \arrow[r] \arrow[d] & k \oplus V_*[1] \arrow[d] \\ k \oplus V'_*[1] \arrow[r] & k \oplus (V_* \oplus V'_*)[1] \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

is pushout in Mod_k, which it is. ∎

C^* preserves small colims,

$$\begin{tikzcd} C^*: \mathrm{Lie}_k \arrow[r] & (\mathrm{CAlg}_k^{aug})^{op} : \mathfrak{D} \arrow[l, bend right] \end{tikzcd}$$

presentable

$\Longrightarrow$ has a right adjoint $\mathfrak{D}$

Remark: $C^*_{dg} : \mathrm{Lie}_k^{dg} \longrightarrow (\mathrm{CAlg}_k^{dg})^{op}_{aug}$

on homotopy categories, C^*_{dg} has a right adjoint,
but C^*_{dg} doesn't (not left Quillen functor)
at level of model categories.

$\rightarrow$ not an easy description of $\mathfrak{D}$ using dgla's.
could use L_∞-algebras instead!

(2) Analyze $\mathfrak{D}$ on certain "good" dgla's $\rightsquigarrow$. $u : \mathfrak{g}_* \longrightarrow \mathfrak{D}C^*(\mathfrak{g}_*)$
is an equivalence!

Recall from Owen:

$$\begin{tikzcd} \mathrm{Lie}_k \arrow[rr, "C^*"'] \arrow[dr, "\theta"'] & & (\mathrm{CAlg}_k^{aug})^{op} \arrow[ll, bend right, "\mathfrak{D}"'] \arrow[dl, "T{[-1]} \circ \mathrm{Spf}"] \\ & \mathrm{Mod}_k \arrow[ul, bend right, "\mathrm{Free}"'] & \end{tikzcd}$$

$$\begin{tikzcd} \mathrm{Mod}_k \arrow[r, "\mathrm{Free}"] & \mathrm{Lie}_k \arrow[l, bend right, "\theta"'] \arrow[r, "C^*"] & (\mathrm{CAlg}_k^{aug})^{op} \arrow[l, bend right, "\mathfrak{D}"'] \end{tikzcd}$$

induced by

$$\mathrm{Vect}^{dg} \longrightarrow \mathrm{CAlg}_k^{dg}$$

$$V \longmapsto C^*(\mathrm{Free}(V)) \underset{\text{Prop 2}}{\xrightarrow{\sim}} k \oplus V^\vee[-1]$$

$$V \longmapsto k \oplus V^\vee[-1]$$

$$\begin{tikzcd} \mathrm{Mod}_k \arrow[rr] \arrow[dr, "D"'] & & (\mathrm{CAlg}_k^{\mathrm{aug}})^{op} \\ & \mathrm{Mod}_k^{op} \arrow[ur] & \end{tikzcd}$$

$D : \mathrm{Mod}_k \longrightarrow \mathrm{Mod}_k^{op}$

from $\mathrm{Vect}_k^{dg} \longrightarrow (\mathrm{Vect}_k^{dg})^{op}$

$V \longmapsto V^\vee[-1]$

$\mathrm{Mod}_k^{op} \longrightarrow (\mathrm{CAlg}_k^{\mathrm{aug}})^{op}$

$M \longmapsto k \oplus M$

square-zero ext.

$V^\vee[1] \leftarrow\!\!\text{- - -}\!\!\mapsto V$

right adjt

"$\mathbb{L}_{A/k} \underset{A}{\otimes} k$" $\leftarrow\!\!\text{- - - -}\!\!\mapsto k \to A \to k$

$\cong$

$\mathbb{L}_{k/A}[-1]$

Prop $\mathfrak{g}_*$ dgLa_k, char $k=0$, if

(a) $\forall n$, $\mathfrak{g}_n$ is a finite dim'l v.sp.

(b) $\mathfrak{g}_n$ is trivial $\forall\, n \geq 0$

$\Longrightarrow$ unit map $u : \mathfrak{g}_* \longrightarrow \mathfrak{D}C^*(\mathfrak{g}_*)$ is an equivalence in Lie_k

Lemma: $(\mathrm{CAlg}_k^{\mathrm{aug}})^{op} \xrightarrow{\mathfrak{D}} \mathrm{Lie}_k \xrightarrow{\theta} \mathrm{Mod}_k$ is given on obj. by

$A \longmapsto \underbrace{\mathbb{L}_{k/A}^\vee}$

$\shortparallel$

$T_0\,\mathrm{Spf}$

Claim: in this situation, $u : \mathfrak{g}_* \longrightarrow \mathfrak{D}C^*(\mathfrak{g}_*)$ in Lie_k

$\downarrow \theta$

$\mathfrak{g}_* \longmapsto \mathfrak{g}_*^{\vee\vee}$ in Mod_k

Strategy: as graded v.sp, $C^*(\mathfrak{g}_*) \simeq \prod_{n \geq 0} (\mathrm{Sym}^n \mathfrak{g}_*[1])^\vee$

Choosing basis for $\mathfrak{g}_{-1}$ $\rightsquigarrow$ dual basis $\{x_1, \dots, x_p\}$ for $\mathfrak{g}_{-1}^\vee$.

$$C^0(\mathfrak{g}_*) \simeq k[[x_1, \dots, x_p]]$$

$$A^*_* := \bigoplus_{n \geq 0} \left(\mathrm{Sym}^n \mathfrak{g}_*[1]\right)^\vee \subset C^*(\mathfrak{g}_*) \quad \text{dg subalg.}$$

$$A_0 \supset k[x_1, \dots, x_p]$$

Assumptions (a) + (b) $\Longrightarrow$ A_* is graded polynomial ring generated by $\mathfrak{g}_*^\vee[-1]$,

$$A_* \otimes_{k[x_1, \dots, x_p]} k[[x_1, \dots, x_p]] \xrightarrow{\simeq} C^*(\mathfrak{g}_*)$$

iso of cdga's

$\rightsquigarrow$ Yoga of relative cotangent complexes.