

The main theorem cont.

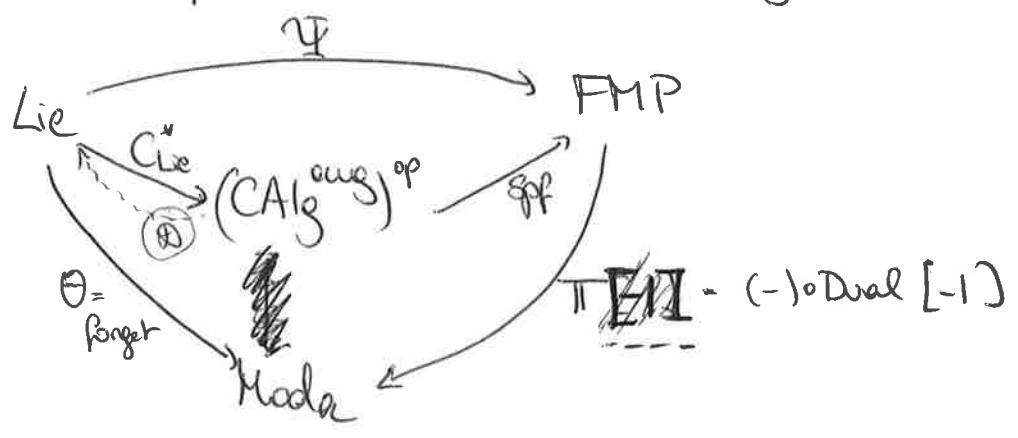
COAL: $C_{Lie}^* : Lie \longrightarrow (CAlg^{aug})^{op} = \mathcal{D}$

(1) C_{Lie}^* has right adjoint \mathcal{D}

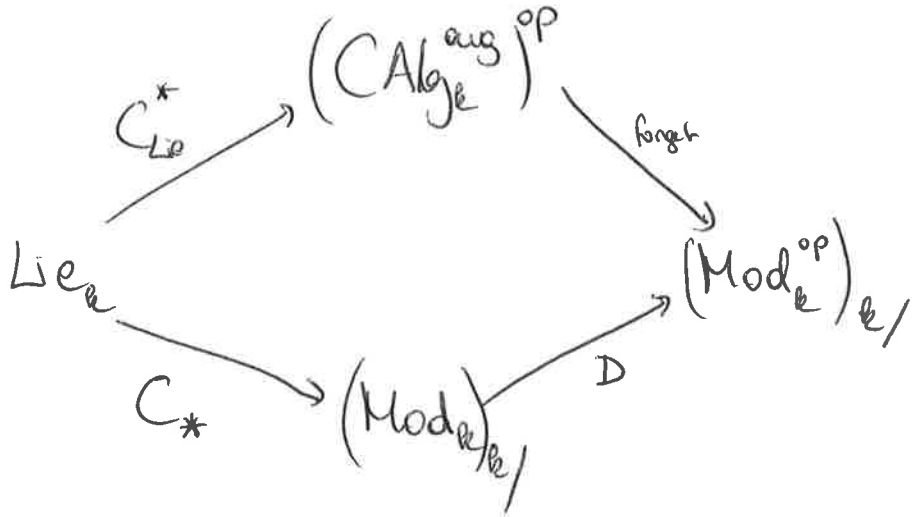
use: adjoint functor theorem

→ need to show that C_{Lie}^* preserves small colims

(2) Then show: this adjoint is "almost" an equivalence, ~~with these indices~~
 ⇒ the equivalence Ψ we ultimately want.



(1) C_{Lie}^* preserves small colims.



•) D induced by $\left\{ \begin{array}{l} Vect_k^{dg} \longrightarrow (Vect_k^{dg})^{op} \\ V_* \longmapsto V_*^v \end{array} \right.$ which sends homotopy colims to homotopy co lims $\Rightarrow D$ preserves small colims.

•) colimits in $(CAlg_k^{aug})^{op}$ are computed in $(Mod_k^{op})_{k/}$
 $\Rightarrow C_{Lie}^*$ preserves small colims iff $forget \circ C_{Lie}^*$ does

\Rightarrow enough to show that $C_* : Lie_k \longrightarrow (Mod_k)_{k/}$ preserves small colims. Use:

(A) Prop: $F: \mathcal{C} \rightarrow \mathcal{D}$ preserves small colims iff

(Combining HIT 4.2.3.11 + HA 1.3.3.10) F preserves finite co products + small sifted colimits

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Recall: $C_*(g_*) \stackrel{\text{gr vsp.}}{\cong} \text{Sym}(g_*[1])$, $d(x_{i_1} \dots x_{i_n}) =$
 $\sum (-1)^{\dots} x_{i_1} \dots x_{i_{r-1}} d x_{i_r} x_{i_{r+1}} \dots x_{i_n}$
 $+ \sum (-1)^{\dots} x_{i_1} \dots x_{i_{r-1}} x_{i_{r+1}} \dots x_{j-1} [x_i, x_j] x_{j+1} \dots x_n$

filtration: $\text{Sym}^{\leq n}(g) \cong \bigoplus_{i \leq n} \text{Sym}^i(g)$

$\leadsto k \cong C_*^{\leq 0}(g) \hookrightarrow C_*^{\leq 1}(g) \hookrightarrow C_*^{\leq 2}(g) \hookrightarrow \dots$

$\text{Sym}^n(g) \cong C_*^{\leq n}(g) / C_*^{\leq n-1}(g)$ in dg Vect
 \uparrow
 $[1]?$

Prop 1 $f: g_* \rightarrow g'_*$ quasi-iso of dgla's. Then

$C_*(f): C_*(g_*) \rightarrow C_*(g'_*)$ is quasi-iso of chain complexes

Pf: quasi-isos closed under filtered colimits

\Rightarrow Show that $f_n: C_*^{\leq n}(g_*) \rightarrow C_*^{\leq n}(g'_*)$ is q-i. $\forall n \geq 0$

Use induction: $n=0 \checkmark$ $n>0 \rightsquigarrow$ comm. diag of short ex. seq.

$$\begin{array}{ccccccc} 0 & \rightarrow & C_*^{\leq n-1}(g_*) & \rightarrow & C_*^{\leq n}(g_*) & \rightarrow & \text{Sym}^n(g_*[1]) \rightarrow 0 \\ & & \downarrow f_{n-1} & & \downarrow f_n & & \downarrow \phi \\ 0 & \rightarrow & C_*^{\leq n-1}(g'_*) & \rightarrow & C_*^{\leq n}(g'_*) & \rightarrow & \text{Sym}^n(g'_*[1]) \rightarrow 0 \end{array}$$

\Rightarrow by ind. hypothesis, need to show ϕ is quasi-iso.

ϕ is retract of $g^{\otimes n}[n] \xrightarrow{f^{\otimes n}} g'^{\otimes n}[n] \Rightarrow$ quasi-iso.

$\Rightarrow C_*: \text{Lie}_k \rightarrow (\text{Mod}_k)_k /$ functor of ∞ -cats.

Recall: want to show this preserves small colimits.

Step 1

Analyze on free dgla's: ~~... general case to free~~

Prop 2 V_* dg vect. sp., $g_* = \text{Free}_{\text{Lie}}(V_*)$. Then incl. of chain complexes

$$\xi: \mathbb{k} \oplus V_*[1] \hookrightarrow \mathbb{k} \oplus g_*[1] \simeq C_*^{\text{Lie}}(g) \hookrightarrow C_*(g_*)$$

is a quasi-iso.

Recall from Dan ^{+ Matthew's} talks:

$$\begin{aligned} \text{If } A &= \text{Tens}(V) = \text{Tens}(V)/(R) &= U(\text{Free}_{\text{Lie}}(V)) \\ A' &= \text{Tens}(V^{\circ}) / (R^{\perp}) & \\ \parallel & & \\ C_{\text{Lie}}^*(\text{Free}(V)) & & \\ \text{Free}(Ug^i = C_*^{\text{Lie}}(g)) & & \end{aligned}$$

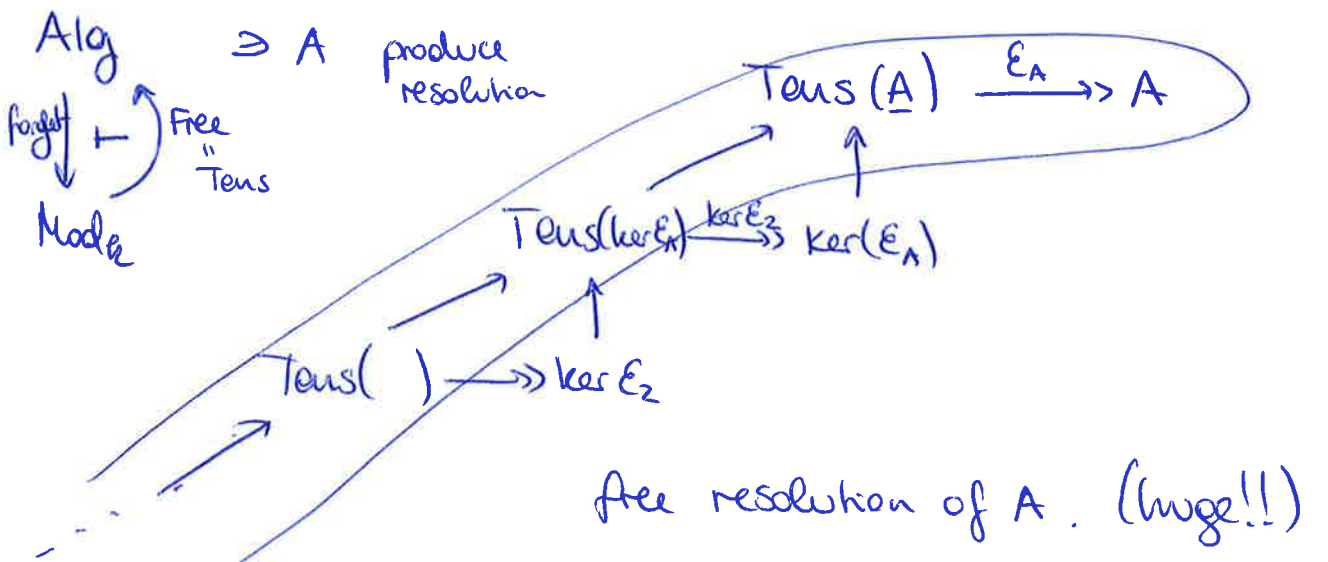
univ. env. alg.

Step 2

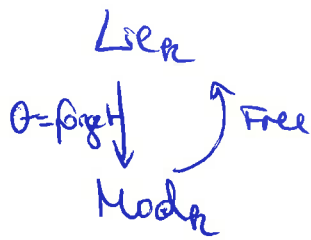
reduce to free case by "resolving"

Claim: g any dgla. Then g is the geometric realization of a simplicial object (g_n) in $\text{Lie}_{\mathbb{k}}$ st. each g_n lies in the essential image of $\text{Free}: \text{Mod}_{\mathbb{k}} \rightarrow \text{Lie}_{\mathbb{k}}$.

Idea:



Similar,



θ is "monadic"

$$T = \theta \circ \text{free monad}$$

\leadsto can do monadic resolution in ∞ -categorical setting.

Prop $C_*: \text{Lie}_k \rightarrow (\text{Mod}_k)_{k/}$ preserves small colimits.

Pf: Show: (a) preserves small sifted colimits
(b) \otimes finite coproducts

(small) (a) sifted colims: (of filtered colim + geometr. realizations)



Strategy similar to previous prop 1

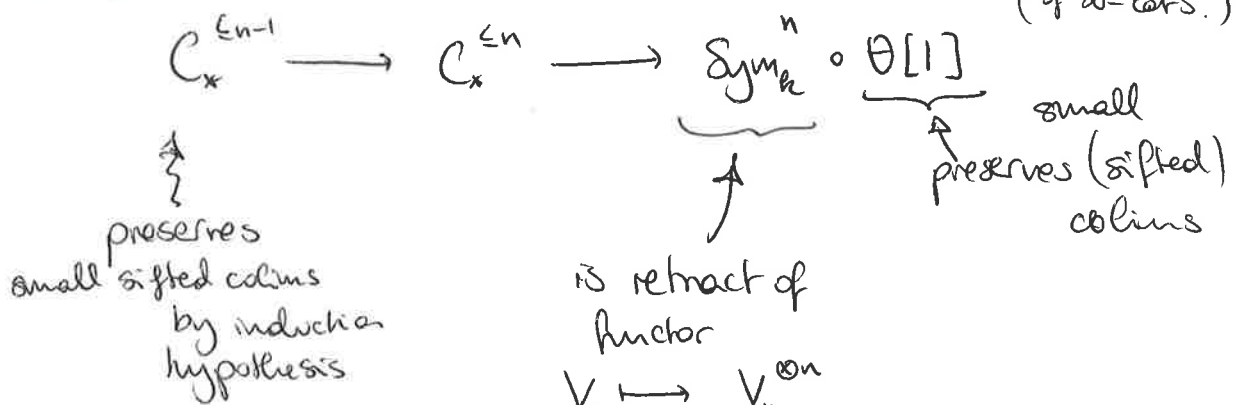
$$\forall n: C_*^{\leq n}: \text{Lie}_k \rightarrow \text{Mod}_k,$$

C_* is filtered colimit of these functors!

\implies HIT 5.5.2.3 It suffices to show that $C_*^{\leq n}$ preserve sifted colims.

Induction on n: $n \leq 0$ trivial.

~~next~~ induction step: Have fiber sequence of functors (of ∞ -cats.)



Here sifted is crucial: eg $n=2$: $(V, W) \xrightarrow{\otimes} V \otimes W$ pres. colims in each var.

$$V \xrightarrow{\text{diag}} (V, V) \xrightarrow{\otimes} V \otimes V$$

eg n=2:

$$\begin{aligned} (\text{Vect}^{\text{dg}})^{\times 2} &\xrightarrow{-\otimes-} \text{Vect}^{\text{dg}} \\ (V, W) &\longmapsto V \otimes W \end{aligned}$$

pres. colims in each variable

$$\begin{array}{ccccc} \text{Mod}_k \text{Vect}^{\text{dg}} & \xrightarrow{\text{diag}} & \text{Mod}_k \text{Vect}^{\text{dg}}^{\times 2} & \xrightarrow{-\otimes-} & \text{Mod}_k \text{Vect}^{\text{dg}} \\ V & \longmapsto & (V, V) & \longmapsto & V \otimes V \end{array}$$

sifted colim:

$$\begin{array}{ccccc} \text{Mod}_k \text{Vect}^{\text{dg}} & \xrightarrow{\text{diag}} & \text{Mod}_k \text{Vect}^{\text{dg}}^{\times 2} & \xrightarrow{-\otimes-} & \text{Mod}_k \text{Vect}^{\text{dg}} \\ \uparrow & & \uparrow & & \\ k & \xrightarrow{\text{copinal}} & k \times k & & \end{array}$$

sifted diagram ↗

(b) finite coproducts:

- C_* preserves initial objects \implies enough to show that C_* preserves pairw. coprod, i.e. for

$$g_*'' = g_* \amalg g_*'$$

$$\begin{array}{ccc} k & \longrightarrow & C_*(g_*) \\ \downarrow & & \downarrow \\ C_*(g_*') & \longrightarrow & C_*(g_*'') \end{array} \text{ is pushout. in Mod}_k$$

Use "monadic" resolution:

$$\begin{array}{ccc} \text{Lie}_k & \xrightarrow{\text{Free}} & \text{Free} \\ \downarrow \theta & & \downarrow \\ \text{Mod}_k & & \end{array}$$

Write $g_* =$ geometric realization of simpl. object s.t. each $(g_*)_n$ is in essential image of Free

$$g_*' = \text{---} \amalg \text{---} (g_*')$$

$$g_*'' = \text{---} \amalg \text{---} (g_*''). \quad (g_*'')_n = (g_*')_n \amalg (g_*'')_n$$

b/c geom. real commutes w/ coproducts

by (a), C_v commutes w/ geometr. realiz.,
 (b/c are sifted colims)

\Rightarrow enough to show that

$$\begin{array}{ccc}
 k & \longrightarrow & C_v(g_v)_n \\
 \downarrow & & \downarrow \\
 C_v(g'_v)_n & \longrightarrow & C_v(g''_v)_n
 \end{array}$$

is pushout
in Mod_k

free \Rightarrow Show for
 $g_v \simeq \text{Free}(V_v)$
 $g'_v \simeq \text{Free}(V'_v) \Rightarrow g''_v \simeq \text{Free}(V_v \oplus V'_v)$

Prop 2 \Rightarrow enough to show

$$\begin{array}{ccc}
 k & \longrightarrow & k \oplus V_v[1] \\
 \downarrow & & \downarrow \\
 k \oplus V'_v[1] & \longrightarrow & k \oplus (V_v \oplus V'_v)[1]
 \end{array}$$

is pushout in Mod_k ,
 which it is. \square

$C_{\#}^*$ preserves ~~small~~ small colims,

$$C_{\#}^*: \text{Lie}_k \xrightarrow{\quad} (\text{CA}lge_k^{\text{aug}})^{\text{op}} \cong \mathcal{D}$$

\uparrow
 presentable

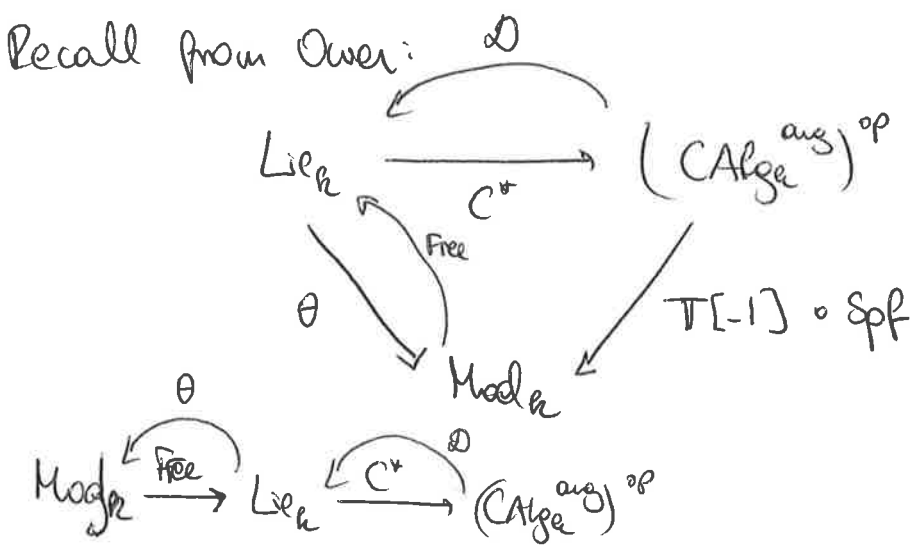
\Rightarrow has a right adjoint \mathcal{D}

Remark: $C_{\text{dg}}^*: \text{Lie}_k^{\text{dg}} \rightarrow (\text{CA}lge_k^{\text{dg}})^{\text{op}}$

on homotopy categories, C_{dg}^* has a right adjoint, but C_{dg}^* doesn't (not left Quillen functor) at level of model categories.

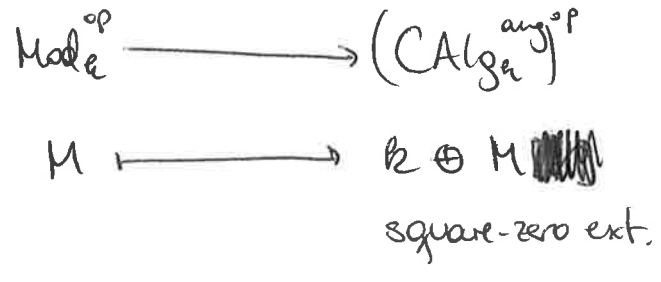
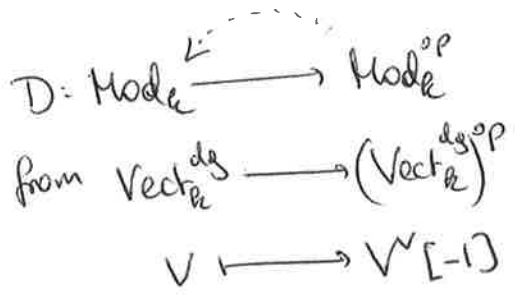
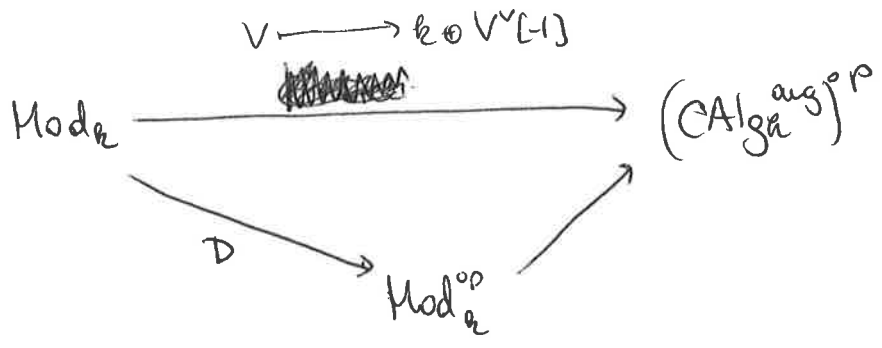
\rightarrow not an easy description of \mathcal{D} using dglas. could use L_{\infty}-algebras instead!

(2) Analyze \mathcal{D} on certain "good" dglas \rightarrow ~~get~~ ~~is~~ ~~an~~ ~~equivalence!~~
~~is an equivalence!~~ $v: g_* \rightarrow \mathcal{D}C^*(g_*)$ is an equivalence!



induced by

$$\begin{array}{ccc}
 \text{Vect}^{\text{ds}} & \longrightarrow & \text{CA}lge_k^{\text{ds}} \\
 v & \longrightarrow & C^*(\text{Free}(v)) \xrightarrow{\cong} k \oplus V^V[-1] \\
 & & \text{Prop 2}
 \end{array}$$



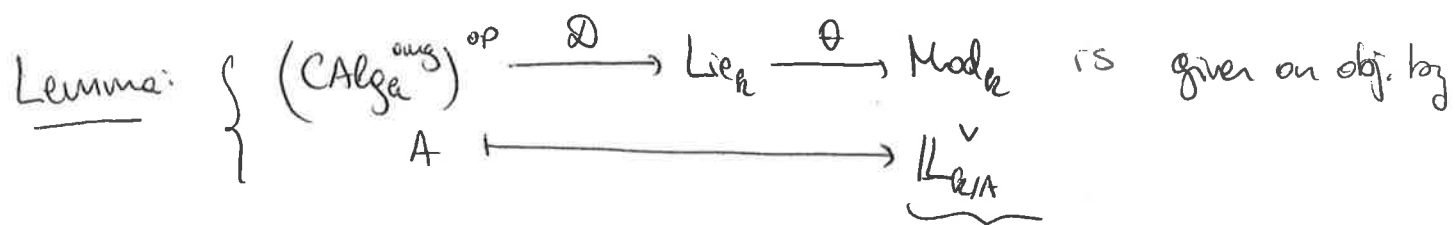
$$V[-1] \xleftarrow{\quad \quad} V \text{ right adjt}$$

$$\begin{array}{c}
 \text{" } \mathbb{L}_{A/k} \otimes_k^{\mathbb{L}} k \text{ " } \xleftarrow{\quad \quad} k \rightarrow A \rightarrow k \\
 \parallel \\
 \mathbb{L}_{k/A}[-1]
 \end{array}$$

Prop g_* dgl $_k$, char $k=0$, if

- (a) $\forall n, g_n$ is finite dim'l v.sp.
- (b) g_n is trivial $\forall n \geq 0$

\implies univ map $u: g_* \rightarrow \mathcal{D}C^*(g_*)$ is an equivalence in Lie_k



Claim: in this situation, $u: g_* \rightarrow \mathcal{D}C^*(g_*)$ in Lie_k

$$g_* \xrightarrow{\quad \quad} g_*^{\vee \vee} \text{ in } \text{Mod}_k$$

Strategy: as graded v.sp, $C^*(g_*) \simeq \prod_{n \geq 0} (\text{Sym}^n g_*[1])^{\vee}$

Choosing basis for $\mathfrak{g}_- \rightsquigarrow$ dual basis $\{x_1, \dots, x_p\}$ for \mathfrak{g}_-^\vee . 10

$$C^0(\mathfrak{g}_*) \simeq k[x_1, \dots, x_p]$$

$$A_*^\# := \bigoplus_{n \geq 0} (\text{Sym}^n \mathfrak{g}_*[-1])^\vee \subset C^*(\mathfrak{g}_*) \quad \text{dg subalg.}$$

$$A_0 \supset k[x_1, \dots, x_p]$$

Assumptions \implies A_* is graded polynomial ring generated by $\mathfrak{g}_*^\vee[-1]$,
 (a) + (b)

$$A_* \otimes_{k[x_1, \dots, x_p]} k[x_1, \dots, x_p] \xrightarrow{\cong} C^*(\mathfrak{g}_*)$$

iso of
cdga's

\rightsquigarrow Yoga of relative cotangent complexes.